New Type of Difference Sequence Space of Fibonacci Numbers

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Abstract

The aim of the present paper is to introduce a new band matrix \( H_u \) and define the sequence space

\[
\ell_p(\mathcal{H}_u) = \left\{ \sum_k \left| \frac{f_k}{f_{k+1}} u_k x_k - \frac{f_k}{f_{k+1}} u_{k-1} x_{k-1} \right|^p \right\} \quad 1 \leq p \leq \infty
\]

where \( f_k \) is the \( k \)th Fibonacci number for every \( k \in \mathbb{N} \). We study some topological properties of this space and also some inclusion relations involving these spaces are introduced.

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Keywords

Sequence spaces; Fibonacci numbers; BK-spaces

1. Preliminaries Background and Notation

A sequence space is defined to be a linear space of real or complex sequences. Throughout the paper, \( \mathbb{N}, \mathbb{R} \) and \( \mathbb{C} \) denote the set of non-negative integers, the set of real numbers and the set of complex numbers, respectively. Let \( \omega \) denote the space of all sequences (real or complex); \( l_\infty, c \) and \( c_0 \) respectively, denotes the space of all bounded sequences, the space of convergent sequences and space of null sequences. Also, by \( l_1 \) and \( l_p \) \( (1 < p < \infty) \), we denote the spaces of all absolutely convergent and \( p \)-absolutely convergent series, respectively.

A sequence space \( X \) is called an \( FK \)-space if it is a complete linear metric space with continuous coordinates \( p_x: X \to \mathbb{C} \) defined by \( p_x(x) = x \) for all \( x \in X \) and every \( n \in \mathbb{N} \). A \( BK \)-space is a normed \( FK \)-space, that is, a \( BK \)-space is a Banach space with continuous coordinates [1-4].

The phrase Fibonacci numbers refers to a sequence of numbers studied by a man named Leonardo of Pisa, who was nicknamed “Fibonacci”. The Fibonacci numbers are the sequence of numbers \( \{f_n\}, n \in \mathbb{N} \) defined by recurrence relations

\[ f_0 = 0, f_1 = 1 \text{ and } f_n = f_{n-1} + f_{n-2}; \quad n \geq 2 \]

First derived from the famous rabbit problem of 1228, the Fibonacci numbers were originally used to represent the number of pairs of rabbits born of one pair in a certain population. Let us assume that a pair of rabbits is introduced into a certain place in the first month of the year. This pair of rabbits will produce one pair of offspring every month, and every pair of rabbits will begin to reproduce exactly two months after being born. No rabbit ever dies and every pair of rabbits will reproduce perfectly on schedule see Figure 1 below.

So, in the first month, we have only the first pair of rabbits. Likewise, in the second month, we again have only our initial pair of rabbits. However, by the third...
month, the pair will give birth to another pair of rabbits, and there will now be two pairs. Continuing on, we find that in month four we will have 3 pairs, then 5 pairs in month five, then 8, 13, 21, 34, ..., etc, continuing in this manner. It is quite apparent that this sequence directly corresponds with the Fibonacci sequence introduced above, and indeed, this is the first problem ever associated with the now-famous numbers.

**Figure 1:** Rabbit-Breeding Tree

<table>
<thead>
<tr>
<th>Month</th>
<th>Pairs</th>
<th>Number of pairs of adults (A)</th>
<th>Number of pairs of babies (B)</th>
<th>Total pairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>A</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>February</td>
<td>A</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>March</td>
<td>B</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>April</td>
<td>A</td>
<td>3</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>May</td>
<td>B</td>
<td>5</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>June</td>
<td>A</td>
<td>8</td>
<td>5</td>
<td>13</td>
</tr>
<tr>
<td>July</td>
<td>B</td>
<td>13</td>
<td>8</td>
<td>21</td>
</tr>
<tr>
<td>August</td>
<td>A</td>
<td>21</td>
<td>13</td>
<td>34</td>
</tr>
<tr>
<td>September</td>
<td>B</td>
<td>34</td>
<td>21</td>
<td>55</td>
</tr>
<tr>
<td>October</td>
<td>A</td>
<td>55</td>
<td>34</td>
<td>89</td>
</tr>
<tr>
<td>November</td>
<td>B</td>
<td>89</td>
<td>55</td>
<td>144</td>
</tr>
<tr>
<td>December</td>
<td>A</td>
<td>144</td>
<td>89</td>
<td>233</td>
</tr>
<tr>
<td>January</td>
<td>B</td>
<td>233</td>
<td>144</td>
<td>377</td>
</tr>
</tbody>
</table>

The Fibonacci sequence itself can be explained like this: each number in the sequence is the sum of the two previous numbers in the sequence. Therefore, the sequence begins with 0, and then continues on like this: 1, 1, 2, 3, 5, 8, etc. The higher up the chain you go, the consecutive numbers would also then be divisible by each other to obtain what called the “golden ratio” that became a staple in Renaissance painting proportions. The ratio works out to be 1:1.618 which painters used to proportion their work because they believed it to be much more aesthetically pleasing. This same ratio is found time and again in nature as well, making it fascinating for scientists to study the sequence organically see, [5-8]. Some basic properties are as follows:

\[
\sum_{k=0}^{n} f_k = f_{n+2} - 1; n \in \mathbb{N},
\]

and

\[
\sum_{k=0}^{n} f_k^n = f_{n+1}; n \in \mathbb{N}
\]

Kizmaz [9] defined the difference sequence spaces \( Z(\Delta) \) as follows

\( Z(\Delta) = \{ x = (x_i) \in \omega: (\Delta x_i) \in Z \} \)

where, \( Z \in \{ l_\infty, c, c_0 \} \) and \( \Delta x_k = x_k - x_{k+1} \).
Basar, Altay and Mursaleen [10] has studied the sequence space \( b_{\Delta}^p(u) \) as
\[
b_{\Delta}^p(u) = \{ x = (x_k) \in \omega : \sum_k |u_k \Delta x_k|^p < \infty \} ,
\]
where \( 1 \leq p < \infty \) alternatively, the space \( b_{\Delta}^p(u) \) can be redefined as
\[
b_{\Delta}^p(u) = (l_p)\Delta^u, 1 \leq p < \infty
\]
where \( \Delta^u = \{ \Delta_{nk}^u \} \) with
\[
\Delta_{nk}^u = (-1)^{n-k} u_k, \quad \text{if } n-1 \leq k \leq n,
\]
\[
\text{if } 0 \leq k \leq n-1 \text{ or } k > n.
\]

2. The Fibonacci Difference Sequence Space

\( l_p\left( H_u \right) \) In the present section of the paper, we introduce a new type of difference sequence space \( l_p(\Delta_u) \) by using Fibonacci numbers \( f_n \) and show it is linearly isomorphic to \( l_p^v \).

Hamid et al. [4] introduce the so called \( H \)-matrix generated by Fibonacci numbers as follows
\[
h_{nk}^u = \begin{cases} \frac{u_k f_k^2}{f_n f_{n+1}} & \text{if } 0 \leq k \leq n, \\ 0 & \text{if } k > n. \end{cases}
\]

Note that if we take \( q^k = f_k^2 \), then the matrix \( H \) is special case of the matrix \( R_u^q \left( Q_n = \sum_{k=0}^n f_k^2 = f_n f_{n+1} \right) \) introduced by Neyaz and Hamid [4].

For \( X \) to be any of the spaces \( l_\infty, c, c_0 \) and \( l_p (1 \leq p < \infty) \), Hamid et al [6] introduced the Fibonacci sequence space \( X(\tilde{H}_u) \) and is defined by
\[
X(\tilde{H}_u) = \{ x = (x_k) \in \omega : y = (y_k) \in X \}
\]
where the sequence \( y = (y_k) \) is the \( H \)-transform of the sequence \( x = (x_k) \) and is given by
\[
y_k = H_k(x) = \frac{1}{f_k f_{k+1}} \sum_{i=0}^k f_i^2 u_i x_i \quad \text{for all } k \in \mathbb{N} \quad (2.1)
\]
Following, Kizmaz [7], Hamid et al [11, 6, 12], Mursaleen and Noman [2], we now define \( l_p\left( \tilde{H}_u \right) \) and \( l_\infty\left( \tilde{H}_u \right) \) as the set of all sequences such that their \( \tilde{H}_u \)-transforms are in the space \( l_p \) and \( l_\infty \), respectively, i.e.,
\[
l_p\left( \tilde{H}_u \right) = \{ x = (x_k) \in \omega : \sum_k \frac{|f_k u_k x_k - f_k u_{k-1} x_{k-1}|^p}{f_{k+1}} \}, 1 \leq p < \infty,
\]
\[
l_\infty\left( \tilde{H}_u \right) = \{ x = (x_k) \in \omega : \sup_k \frac{|f_k u_k x_k - f_k u_{k-1} x_{k-1}|}{f_{k+1}} < \infty \}.
\]
Alternatively, the sequence spaces \( l_\infty\left( \tilde{H}_u \right) \) and \( l_\infty\left( \tilde{H}_u \right) \) many be redefined
\[
l_p\left( \tilde{H}_u \right) = (l_p)_{\tilde{H}_u} \quad \text{and} \quad l_\infty\left( \tilde{H}_u \right) = (l_\infty)_{\tilde{H}_u} \quad (2.2)
\]
Define the sequence $y = (y_k)$, which will be frequently used, by the $H_u$ transform of a sequence $x = (x_k)$, i.e.,

$$y_k = \begin{cases} \frac{f_0}{u_0} x_0 & \text{if } k = 0 \\ \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} u_{k-1} x_{k-1} & \text{if } k \geq 1. \end{cases}$$  \hspace{1cm} (2.3)

Now, we may begin with the following theorem which is essential in the text.

**Theorem 2.1**

Let $1 \leq p \leq \infty$. Then $l_p(H_u)$ is a BK-space with the norm given by

$$||x||_{l_p(H_u)} = \left( \sum_{k} |y_k|^p \right)^{1/p} \quad (1 \leq p < \infty),$$

and

$$||x||_{l_\infty(H_u)} = \sup_{k} |y_k|.$$

**Proof**

Since (2.2) holds and $l_p$ and $l_\infty$ are BK-spaces with respect to their usual norms see, [6, 8] and the matrix $H_u$ is a triangle; Theorem 4.3.2 of Wilansky [13] gives the fact that $l_p(H_u)$ and $l_\infty(H_u)$ are BK-spaces with the given norms, where $1 \leq p \leq \infty$.

**Remark**

One can easily check that the absolute property does not hold on the spaces $l_p(H_u)$ and $l_\infty(H_u)$, that is $||x||_{l_p(H_u)} \neq |||x|||_{l_p(H_u)}$ and $||x||_{l_\infty(H_u)} \neq |||x|||_{l_\infty(H_u)}$ for atleast one sequence in the spaces $l_p(H_u)$ and $l_\infty(H_u)$, and this shows that $l_p(H_u)$ and $l_\infty(H_u)$ are the sequence spaces of non-absolute type, where $|x| = (|x|)$.

**Theorem 2.2**

The space $l_p(H_u)$ of non-absolute type is linearly isomorphic to the space $l_p$, i.e., $l_p(H_u) \cong l_p$ for $1 \leq p \leq \infty$.

**Proof**

To prove the result we should show the linear bijection between the spaces $l_p(H_u)$ and $l_p$ for $1 \leq \infty$. For that consider the transformation $T$ from $l_p(H_u)$ to $l_p$ by $x \rightarrow y = Tx$. Then, the linearity of $T$ is trivial. Further, we see that $x = 0$ whenever $Tx = 0$ and consequently $T$ is injective.

Moreover, let $y = (y_k) \in l_p$; for $1 \leq p \leq \infty$ be given and define the sequence $x = (x_k)$ by

$$x_k = \sum_{j=0}^{k} \frac{f_{j+1}^2 u_j^{-1}}{f_j f_{j+1}} y_j, \quad k \in \mathbb{N}.$$
Then, in the case $1 \leq p \leq \infty$ and $p = \infty$, we get

$$ \|x\|_p(H_u) = \left( \sum_k \left| \frac{f_k}{f_{k+1}} u_k x_k - \frac{f_{k+1}}{f_k} u_{k-1} x_{k-1} \right|^p \right)^{\frac{1}{p}} $$

$$ = \left( \sum_k \sum_i \frac{f_k}{f_{k+1}} \frac{f_{k+1}}{f_i} y_i - \frac{f_{k+1}}{f_k} \sum_{i=0}^k \frac{f_{k+1}}{f_i} y_i \right)^{\frac{1}{p}} $$

$$ = \left( \sum_k |y_k|^p \right)^{\frac{1}{p}} = \|y\|_p < \infty, $$

and

$$ \|x\|_{L_p(H_u)} = \sup_k |y_k| = \|y\|_{\infty} < \infty, $$

respectively. Thus, we have $x \in l_p(H_u)$ ($1 \leq p \leq \infty$). Hence, $T$ is subjective and norm preserving. Consequently, $T$ is linear bijection which shows that the spaces $l_p(H_u)$ and $l_p$ are isometrically isomorphic for $1 \leq p \leq \infty$. Hence, the proof of the result is complete.

**Theorem 2.3**

The inclusion $l_p \subset l_p(H_u)$ strictly holds for $1 \leq p \leq \infty$.

**Proof**

To prove the validity of inclusion $l_p \subset l_p(H_u)$ for $1 \leq p \leq \infty$, it is suffice to show the existence of a number $M > 0$ such that $\|x\|_p(H_u) \leq M \|x\|_p$ for every $x \in l_p$.

Let $x \in l_p$ and $1 \leq p \leq \infty$. Since the inequalities $\frac{f_k}{f_{k+1}} \leq 1$ and $\frac{f_{k+1}}{f_k} \leq 2$ hold for every $k \in \mathbb{N}$, we obtain with the notation of (2.3)

$$ \sum_k |y_k|^p \leq \sum_k 2^{p-1} (|x_k|^p + |2x_{k-1}|^p) \leq 2^{p-1} \left( \sum_k |x_k|^p + \sum_k |x_{k-1}|^p \right) $$

and

$$ \sup_k |y_k|^p \leq 3 \sum_k |x_k| $$

which together yields, as expected,

$$ \|x\|_p(H_u) \leq 4 \|x\|_p $$

For $1 \leq p \leq \infty$. Further, let $u = e (1, 1, 1, \ldots)$ and since the sequence $x = (x_k) = (f_k^2) = (1, 2^2, 3^2, 5^2, \ldots)$ is
in $l_p(\tilde{H}_u) - l_p$ the inclusion $l_p \subset l_p(\tilde{H}_u)$ is strict for $1 \leq p \leq \infty$. Similarly, one can easily prove that inequality (2.4) holds in the case $p = 1$, and so we omit the proof. This completes the proof.

**Theorem 2.4**

Neither the space $bv_p(u)$ and $l_p(\tilde{H}_u)$ include the other for $1 \leq p \leq \infty$.

**Proof**

The proof follows from the following example:

Let $u = e = (1, 1, 1, \ldots)$ and $x = (x_k)(f_{k+1}^2)$. Then, since $\tilde{H}_u$ transform of $x$ is $(1, 0, 0, \ldots)$

which is in $l_p$ and $\Delta x = (1, f_0, f_3, f_1, f_4, \ldots, f_{k-1}, f_{k+2}, \ldots)$ $\not\in l_p$, we conclude that $x \in l_p(\tilde{H}_u)$ but not in $bv_p(u)$. Now, consider the equation for $u_k = (1, 1, 1, \ldots)$

$$\frac{f_k}{f_{k+1}} - \frac{f_{k+1}}{f_k} = \frac{f_k^2 - f_{k+1}^2}{f_k f_{k+1}} = \frac{(-1)^k - f_k f_{k+1}}{f_k f_{k+1}}, \quad (k \in \mathbb{N}).$$

Then, $\left|(-1)^k - f_k f_{k+1}\right| > f_k f_{k+1}$ whenever $k$ is odd, which implies that the series $\left|\frac{f_k}{f_{k+1}} - \frac{f_{k+1}}{f_k}\right|^p$ is not convergent,

where $1 \leq p \leq \infty$. Thus, $\tilde{H}_u x = e = \left(\frac{f_k}{f_{k+1}} - \frac{f_{k+1}}{f_k}\right)$ is not in $l_p$. Hence, the space $l_p(\tilde{H}_u)$ and $bv_p(u)$ overlap but neither contains the other, as desired.

**Theorem 2.5**

If $1 \leq p \leq s$, then $l_p(\tilde{H}_u) \subset l_s(\tilde{H}_u)$.

**Proof**

The proof is omitted.

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